Lecture 1: Quantification of Waves

I. Black-body Radiation

In the end of XIX Century one on the most intriguing experiments refers to ability of matter being converted from the thermal to electromagnetic (or spectral) energy through intermediated by "fully" absorbed-emitted object, hereafter called black-body.

The point was referring to analytically describe the obtained spectra by means of the so called spectral density

$$\rho(\upsilon) = \frac{N(\upsilon)\langle E\rangle}{V} \quad (1.1)$$

$$N(\upsilon) = \frac{dN_\upsilon}{d\upsilon} \quad (1.)$$

$$N_\upsilon = \frac{\frac{1}{8}\left(\frac{4\pi}{3}\upsilon^3\right)}{\frac{1}{2}\left(\frac{c}{2L}\right)^3} = \frac{8\pi L^3\upsilon^3}{3c^3} \quad (1.)$$

$$N(\upsilon) = \frac{8\pi L^3\upsilon^2}{c^3} \quad (1.)$$

$$\langle E\rangle = \frac{\int_0^\infty E\,\wp(E)dE}{\int_0^\infty \wp(E)dE} \quad (1.)$$

$$\wp(E) = \frac{1}{k_B T}\exp\left(-\frac{E}{k_B T}\right) \quad (1.)$$

$$\int_0^\infty \wp(E)dE = 1 \quad (1.)$$

$$\langle E\rangle_T = \int_0^\infty E\,\wp(E)dE$$

$$= -\int_0^\infty E\frac{\partial}{\partial E}\left[\exp\left(-\frac{E}{k_B T}\right)\right]dE$$

$$= -\int_0^\infty \left\{\frac{\partial}{\partial E}\left[E\exp\left(-\frac{E}{k_b T}\right)\right] - \exp\left(-\frac{E}{k_b T}\right)\right\}dE$$

$$= -\left[E\exp\left(-\frac{E}{k_b T}\right)\right]_0^{\infty} + k_B T = k_B T \text{ (1.)}$$

Alternatively, one can directly compute the thermal energy average as:

$$\langle E\rangle_{\beta=1/k_B T} = \frac{\int_0^{\infty} E\exp(-E\beta)dE}{\int_0^{\infty}\exp(-E\beta)dE} = -\frac{d}{d\beta}\left[\log\int_0^{\infty}\exp(-E\beta)dE\right] = \frac{1}{\beta} = k_B T \quad \text{(1.)}$$

$$\rho_T(\upsilon) = \frac{8\pi k_B T\upsilon^2}{c^3} \equiv \rho_{R-J}(\upsilon) \text{ (1.)}$$

II. Planck's Law

$$E_{Planck} = E(\upsilon) = nh\upsilon \text{ (1.)}$$

$$\langle E\rangle_{\upsilon} = \frac{\sum_{n=0}^{\infty} E(\upsilon)\wp(E(\upsilon))}{\sum_{n=0}^{\infty}\wp(E(\upsilon))} \quad \text{(1.)}$$

$$\langle E\rangle_{\upsilon} = \frac{\sum_{n=0}^{\infty}\frac{nh\upsilon}{k_B T}\exp\left(-\frac{nh\upsilon}{k_B T}\right)}{\sum_{n=0}^{\infty}\frac{1}{k_B T}\exp\left(-\frac{nh\upsilon}{k_B T}\right)} \quad \text{(1.)}$$

$$x = \frac{h\upsilon}{k_B T} \quad \text{(1.)}$$

$$\langle E\rangle_{\upsilon} = h\upsilon\frac{\sum_{n=0}^{\infty} nx\exp(-nx)}{\sum_{n=0}^{\infty} x\exp(-nx)} \quad \text{(1.)}$$

$$\sum_{n=0}^{\infty} nx\exp(-nx) = xe^{-x}\left[1 + 2e^{-x} + 3\left(e^{-x}\right)^2 + \ldots\right] = \frac{xe^{-x}}{\left(1-e^{-x}\right)^2} \quad \text{(1.)}$$

$$f(y) = 1 + 2y + 3y^2 + \ldots \text{(1.)}$$

$$y = \exp(-x) \text{ (1.)}$$

$$g(y)=\int_0^y f(y)dy \Rightarrow f(y)=\frac{d}{dy}g(y) \text{ (1.)}$$

$$g(y)=y\left(1+y+y^2+...\right)=\frac{y}{1-y} \text{ (1.)}$$

$$f(y)=\frac{1}{(1-y)^2} \text{ (1.)}$$

$$\sum_{n=0}^{\infty} x\exp(-nx)=x\left[1+e^{-x}+\left(e^{-x}\right)^2+...\right]=\frac{x}{1-e^{-x}} \text{ (1.)}$$

$$\langle E\rangle_{\upsilon}=\frac{h\upsilon}{\exp\left(\frac{h\upsilon}{k_B T}\right)-1} \text{ (1.)}$$

Again, alternatively, one can directly compute the thermal frequency dependent energy average as:

$$\langle E\rangle_{\upsilon,\beta=1/k_BT}=\frac{\sum_{n=0}^{\infty} nh\upsilon\exp(-nh\upsilon\beta)}{\sum_{n=0}^{\infty}\exp(-nh\upsilon\beta)}=-\frac{d}{d\beta}\left[\log\sum_{n=0}^{\infty}\exp(-nh\upsilon\beta)\right]$$

$$=-\frac{d}{d\beta}\left[\log\frac{1}{1-\exp(-h\upsilon\beta)}\right]=\frac{h\upsilon}{\exp(h\upsilon\beta)-1} \text{ (1.)}$$

$$\rho_{Planck}(\upsilon)=\frac{8\pi h}{c^3}\frac{\upsilon^3}{\exp\left(\frac{h\upsilon}{k_B T}\right)-1} \text{ (1.)}$$

$$\upsilon, \frac{h\upsilon}{k_B T}<<1 \Rightarrow \exp\left(\frac{h\upsilon}{k_B T}\right)\cong 1+\frac{h\upsilon}{k_B T} \text{ (1.)}$$

$$\rho_{Planck}(\upsilon)\xrightarrow{\upsilon<<1}\rho_{R-J}(\upsilon) \text{ (1.)}$$

$$\upsilon, \frac{h\upsilon}{k_B T}>>1 \Rightarrow \exp\left(\frac{h\upsilon}{k_B T}\right)-1\cong \exp\left(\frac{h\upsilon}{k_B T}\right) \text{ (1.)}$$

$$\rho_{Planck}(\upsilon)\xrightarrow{\upsilon>>1}0 \text{ (1.)}$$

$$\lim_{h\to 0}\rho_{Planck}(\upsilon)\overset{\frac{0}{0}}{=}\frac{8\pi\upsilon^2}{c^3}\lim_{h\to 0}\frac{\frac{d}{dh}(h\upsilon)}{\frac{d}{dh}\left[\exp\left(\frac{h\upsilon}{k_B T}\right)-1\right]}=\rho_{R-J}(\upsilon) \quad (1.)$$