Lecture 4: De Broglie Wave Packet and Heisenberg Indeterminacy

$$\psi(x,t) = Ae^{i(k_x x - \omega t)} = Ae^{i\alpha} \quad (1.)$$

$$v_{phase} = \frac{dx}{dt} \overset{\alpha = ct.}{=} \frac{\omega}{k_x} = \frac{\hbar\omega}{\hbar k_x} = \frac{E}{p_x} = \frac{mc^2}{mv} = \frac{c^2}{v} > c \ (!?) \quad (1.)$$

Hereafter we will understand that $k_x := k$ so that to simplify the writings. The normalized wave-function

$$\psi(x,t) = \frac{1}{2\Delta k}\int_{-\Delta k}^{\Delta k} A(k)e^{i(kx-\omega t)}dk \quad (1.)$$

$$\Delta k = k - k_0 \,,\ k_0 = 2\pi / \lambda_0$$

$$\begin{cases} \omega \cong \omega_0 + \left(\dfrac{d\omega}{dk}\right)_0 \zeta,\ \zeta = k - k_0 \\ A(k) \cong A(k_0) \end{cases} \quad (1.)$$

$$\psi(x,t) = \frac{A(k_0)e^{i(k_0 x - \omega_0 t)}}{2\Delta k}\int_{-\Delta k}^{+\Delta k} e^{i\left[x - \left(\frac{d\omega}{dk}\right)_0 t\right]\zeta} d\zeta$$

$$= A(k_0)\frac{\sin\left\{\left[x - \left(\dfrac{d\omega}{dk}\right)_0 t\right]\Delta k\right\}}{\left[x - \left(\dfrac{d\omega}{dk}\right)_0 t\right]\Delta k} e^{i(k_0 x - \omega_0 t)}$$

$$\equiv A(x,t)e^{i(k_0 x - \omega_0 t)} \quad (1.)$$

Since considering the limit

$$\lim_{q\to 0}\frac{\sin q}{q} = 1,$$

we have that the maximum of the above wave-function, $\max\psi(x,t)$, is approached under the fulfilling condition

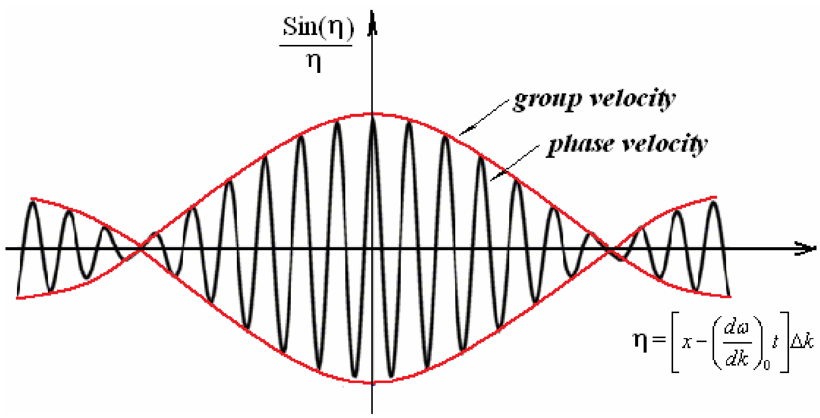
$$x = \left(\frac{d\omega}{dk}\right)_0 t \quad (1.)$$

leading with the so called *group velocity* of the wave packet:

$$v_{group} = \frac{dx}{dt} = \frac{d\omega}{dk} = \frac{dE}{dp} = \frac{d}{dp}\sqrt{m_0^2 c^4 + c^2 p^2} = \frac{pc^2}{E} = v < c \quad (1.)$$

achieving the full significance of a physical velocity; however, it is in an interesting relationship with the phase velocity, i.e.

$$v_{phase} v_{group} = c^2 \quad (1.)$$



Also note that around the point where the group velocity is attached we have the normalization condition preserved at any time:

$$\left|\psi(x,t)\right|_0^2 = A(k_0)^2 = 1 \quad (1.)$$

giving us the indication that the squared wave-function and its squared amplitude are formally equivalent, although as functions of conjugated or reciprocal variables as space and wave-vector, and should be also normalized. The general case will be treated in what follows.

Formal Heisenberg Indeterminacy

$$\psi(x,0) = \frac{1}{\sqrt{2\pi\hbar}} \int_{(\infty)} dp A(p) e^{\frac{i}{\hbar}px} \quad (1.)$$

$$A(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{(\infty)} dx \psi(x,0) e^{-\frac{i}{\hbar}px} \quad (1.)$$

$$\psi(x,0) = \frac{1}{2\pi\hbar} \int_{(\infty)} dp dx \psi(x,0) \quad (1.)$$

$$\Delta p \Delta x \approx 2\pi\hbar = h \quad (1.)$$