Lecture 8: Consequences of Schrödinger Equation

I. Electronic and Photonic Spins

The major consequence of having both Klein-Gordon and Schrödinger equation springs out through looking on the symmetry and dissymmetry with which these equations transforms space in time, respectively. However, in order to achieve the symmetry of space-time transformation in both equations one can rewrite the time-power as being quantified by the s number (hereafter called as *spin*). This way, one can unfold the photonic and electronic wave equation in unitary manner since specializing the Klein-Gordon and Schrödinger equation with common features through imposing the shift $m_0c^2 \to 0$ on the first (regaining the ordinary wave-equation for electromagnetic fields, quantified by photons) and free motion $V(\mathbf{r}) \to 0$ for the second; the resulted photonic and electronic wave equations are:

$$\frac{\partial^{2s(\gamma)}}{\partial t^{2s(\gamma)}}\psi_t(\mathbf{r}) = c^2\nabla^2\psi_t(\mathbf{r}),$$

$$\frac{\partial^{2s(e)}}{\partial t^{2s(e)}}\psi_t(\mathbf{r}) = \frac{i\hbar}{2m_0}\nabla^2\psi_t(\mathbf{r}),$$

from where there follows the *photonic and electronic spin quantization* as:

$$s(\gamma) = 1\ ,\ s(e) = \frac{1}{2}.$$

in order to be in accordance with the operatorial deduced equations, respectively. With this we can affirm that:

- the spin concept is an intrinsic space-time transformation effect;
- the spin quantification depends on the relativistic or non-relativistic level of wave function expression whom the particle evolution is attributed;
- the spin quantification contains, at either relativistic or non-relativistic levels, the quantum influence of motion.

II. Eigen-Functions and Eigen-Values

If once consider appropriate factorized time-coordinate resumed wave-function from above patterned distribution based solution, then the working wave function will look like

$$\psi_t(\mathbf{r}) = \psi_0(\mathbf{r})e^{-\frac{i}{\hbar}Et}$$

while through considering it into the temporal Schrödinger equation it leaves in the first instance with

$$\left[-\frac{\hbar^2}{2m_0}\nabla^2 + V(\mathbf{r})\right]\psi_0(\mathbf{r}) = E\psi_0(\mathbf{r})$$

thus providing the so called stationary Schrödinger equation that in operatorial form simply casts as an eigen-value problem:

$$\hat{H}\psi = E\psi$$

being ψ the eigen-function and E the eigen-value to be determined once the Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m_0}\nabla^2 + V(\mathbf{r})$$

is specialized in terms of applied potential $V(\mathbf{r})$.

In terms of Hamiltonian the corresponding temporal Schrödinger equation would be displayed as generalization of that stationary:

$$\hat{H}\psi = i\hbar\partial_t\psi \ ,$$

where we have introduced the short-notation of time-derivation

$$\partial_t := \frac{\partial}{\partial t} .$$

III. Current Density Probability Conservation Theorem

THEOREM: Schrödinger equation is compatible with charge conservation at the probability density level.

Firstly, let's unfold the meaning of "conservation at the probability density level" in an analytical manner; that is considering a certain volume region of the space, say Γ_Σ, that is characterized by the localization density probability

$$\wp_t(\Gamma_\Sigma) = \int_{\Gamma_\Sigma} |\psi_t(\mathbf{r})|^2 d\mathbf{r}$$

for an electronic containing system with a normalized wave-function:

$$1 = \int_{(\infty)} |\psi_t(\mathbf{r})|^2 d\mathbf{r}$$

remembering that the normalization condition is not depending on time when all space involved.

Therefore, the time-variation of the probability $\wp(\Gamma_\Sigma)$ produces the appearance of the correspondent probability density current according with the basic charge variation-current generation principle

$$\frac{d}{dt}\wp_t(\Gamma_\Sigma) = -\oint_\Sigma \mathbf{j}(\mathbf{r},t)d\boldsymbol{\sigma}_\Sigma$$

where the minus sign means that the diminishing the charge localization in the region Γ_Σ, i.e. the increase of charge probability density outside of this region, is associated with increasing of the appeared current probability density in the complementing region $(\infty \setminus \Gamma_\Sigma)$. Next, the above equation may be rewritten in its equivalent form:

$$\frac{d}{dt}\int_{\Gamma_\Sigma}|\psi_t(\mathbf{r})|^2 d\mathbf{r} = -\oint_\Sigma \mathbf{j}(\mathbf{r},t)d\boldsymbol{\sigma}_\Sigma$$

end even more as

$$\int_{\Gamma_\Sigma}\frac{d}{dt}|\psi_t(\mathbf{r})|^2 d\mathbf{r} = -\int_{\Gamma_\Sigma} div\mathbf{j}(\mathbf{r},t)d\mathbf{r}$$

through the Gauss surface-to-volume integral transformation in the r.h.s. of the last two equations. Thus the pattern charge-current probability density conservation equation casts as:

$$\frac{d}{dt}\rho_t(\mathbf{r}) + div\mathbf{j}(\mathbf{r},t) = 0$$

since recognizing that:

$$\rho_t(\mathbf{r}) = |\psi_t(\mathbf{r})|^2$$

in any region of the space.

With this, the remaining proof of the theorem regards the possibility to regain the above charge-current probability density conservation from the Schrödinger equation. This may be achieved quite straightforward throughout considering both the direct and conjugated temporal Schrödinger equations multiplied by wave-function and conjugated wave-function, in reciprocal manner:

$$-\frac{\hbar^2}{2m_0}\psi^*\nabla^2\psi + V\psi^*\psi = i\hbar\psi^*\partial_t\psi$$

$$-\frac{\hbar^2}{2m_0}\psi\nabla^2\psi^* + V\psi\psi^* = -i\hbar\psi\partial_t\psi^*;$$

Then, their subtraction firstly leads with expression:

$$-\frac{\hbar^2}{2m_0}\left(\psi^*\nabla^2\psi-\psi\nabla^2\psi^*\right)=i\hbar\left(\psi^*\partial_t\psi+\psi\partial_t\psi^*\right)$$

that can be rearranges as:

$$\frac{i\hbar}{2m_0}\nabla\left(\psi^*\nabla\psi-\psi\nabla\psi^*\right)=\partial_t\left(\psi^*\psi\right);$$

Now, recognizing the electronic density probability,

$$\rho_e=(-e)\psi^*\psi$$

and introducing the current density probability as:

$$\mathbf{j}_e=(-e)\mathbf{j}=(-e)\frac{i\hbar}{2m_0}\left(\psi\nabla\psi^*-\psi^*\nabla\psi\right)$$

the wave-function charge conservation law is obtained under the form:

$$\partial_t\rho_e+div\mathbf{j}=0$$

standing as another quantum counter-part for the corresponding electromagnetic law of charge-current conservation.